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Self-consistent screening approximation for critical dynamics

Matteo Campellone† and Jean-Philippe Bouchaud‡

Service de Physique de l'Etat Condensé, CEA-Saclay, Orme des Merisiers, 91 191 Gif s/Yvette CEDEX, France

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Abstract. We generalize Bray's self-consistent screening approximation to describe the critical dynamics of the ϕ^4 theory. In order to obtain the dynamical exponent z, we have to make an ansatz for the form of the scaling functions, which fortunately can be restricted by general arguments. Numerical values of z for $d = 3$, and $n = 1, \ldots, 10$ are obtained using two different ansatz, and differ by a very small amount. In particular, the value of $z \simeq 2.115$ obtained for the three-dimensional Ising model agrees well with recent Monte Carlo simulations.

1. Introduction

Phase-ordering kinetics, critical- and low-temperature dynamics of pure and random systems are the subject of active research [1]. Of particular interest are the approximate methods to deal with nonlinear dynamical equations, which often amount to a self-consistent resummation of perturbation theory [4]. A much-debated case is the 'mode-coupling' approximation, used to describe liquids approaching their frozen (glass) phase. Interestingly, this mode-coupling approximation for systems without disorder can alternatively be seen as the exact equations for an associated *disordered* model of the spin-glass type [2–4]. The simplest mode-coupling approximation for the φ^4 theory, however, is not very good. For example, it predicts for the static critical exponent η the value $2 - \frac{d}{2}$ independently of the number n of components of the field φ. Furthermore, the underlying disordered model is not stable [4].

A better-behaved resummation scheme is the 'self-consistent screening approximation' (SCSA) introduced by Bray in the context of the static φ^4 theory [5, 6], and used in other contexts [7, 8]. It amounts to self-consistently resumming all the diagrams appearing in the large n expansion, including those of order $\frac{1}{n}$. Again, this approximation becomes exact for a particular mean-field-like spin-glass model [4], which turns out to be well defined for all temperatures and thus ensures that the approximation is well behaved.

The aim of the present paper is to generalize the SCSA equations to describe the dynamics of the φ^4 theory *at the critical point*, and to predict a value for the dynamical exponent z.

In section 2 we shall introduce the dynamical SCSA and the dynamical equations in their general form. From section 3 and throughout the rest of the paper we assume that time-translation invariance (TTI) and the fluctuation-dissipation theorem hold at least down

† E-mail address: matteo@amoco.saclay.cea.fr
‡ E-mail address: bouchau@amoco.saclay.cea.fr

to the critical point. Bray's equations will be recovered as the static limit of our dynamical equations. The reliability of the SCSA is discussed quantitatively in the zero-dimensional static case.

In section 4 we study the equations right at the critical temperature where dynamical scaling is supposed to hold. The full solution of these coupled equations, involving *scaling functions*, gives in principle the dynamical exponent z within the SCSA approximation. Unfortunately, as is often the case [9], these equations are very hard to solve, either analytically or even numerically. In sections 5 and 6 we thus propose two different ansatz for the scaling functions, which are much constrained by general requirements, however. The second ansatz leads to the exact $O(\epsilon^2)$ result in the $\epsilon = 4-d$ RG expansion of Halperin *et al* [10]. The numerical value of the exponent z only very weakly depends on the chosen ansatz, and turns out to be quite close to the best available Monte Carlo estimate for the Ising model in $d = 3$[12].

2. The self-consistent screening approximation

Let us consider the coarse-grained Hamiltonian density

$$\mathcal{H}[\varphi(\boldsymbol{x})] = \frac{1}{2}(\nabla\varphi(\boldsymbol{x}))^2 + \frac{\mu}{2}\varphi^2(\boldsymbol{x}) - \frac{g}{8}\varphi^4(\boldsymbol{x}) \tag{2.1}$$

where $\boldsymbol{\varphi}(\boldsymbol{x})$ is an n component field and $\boldsymbol{x}$ is the d-dimensional space variable. With $\boldsymbol{\varphi}^2(\boldsymbol{x})$ and $\boldsymbol{\varphi}^4(\boldsymbol{x})$ we indicate respectively $|\boldsymbol{\varphi}(\boldsymbol{x})|^2$ and $(|\boldsymbol{\varphi}(\boldsymbol{x})|^2)^2$. The coupling constant, g, is negative and of order n^{-1}; μ is a (temperature dependent) mass term which vanishes at the mean-field transition point.

The partition function is

$$Z = \int \mathcal{D}\varphi \mathrm{e}^{-\int \mathrm{d}^d x \frac{\mathcal{H}[\varphi(\boldsymbol{x})]}{T}}. \tag{2.2}$$

In order to introduce the SCSA one starts from a large n expansion formalism. We rewrite Z with a Gaussian transformation introducing an auxiliary field σ

$$Z = \int \mathcal{D}\sigma\mathcal{D}\varphi \mathrm{e}^{-\int \mathrm{d}^d x \frac{\mathcal{H}[\varphi(\boldsymbol{x}),\sigma(\boldsymbol{x})]}{T}} \tag{2.3}$$

$H[\boldsymbol{\varphi}(\boldsymbol{x}), \sigma(\boldsymbol{x})]$ being now the Hamiltonian density of two coupled fields $\boldsymbol{\varphi}(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$.

$$H[\sigma, \varphi] = \frac{1}{2}(\nabla\boldsymbol{\varphi}(\boldsymbol{x}))^2 + \frac{\mu}{2}\boldsymbol{\varphi}^2(\boldsymbol{x}) + \frac{1}{2}\sigma^2(\boldsymbol{x}) - \frac{\sqrt{g}}{2}\sigma(\boldsymbol{x})\boldsymbol{\varphi}^2(\boldsymbol{x}). \tag{2.4}$$

The SCSA amounts to consider the renormalization of the order $1/n$ diagrams in the Dyson expansion for the correlation functions of the two fields $\boldsymbol{\varphi}(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$. Using this resummation scheme Bray [5] obtained interesting results for the static exponent η which describes the small momentum behaviour of the correlation functions. Figure 1 shows the static SCSA equations for $\langle\boldsymbol{\varphi}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}')\rangle$ (full line) and $\langle\sigma(\boldsymbol{x})\sigma(\boldsymbol{x}')\rangle$ (wavy line). The bare quantities are indicated respectively by a thinner plain line and a dashed line.

Our goal is to develop a dynamical generalization of this expansion for non-conserved Langevin dynamics, starting from the SCSA Hamiltonian. We thus obtain the following equations of motion for $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ and $\sigma(\boldsymbol{x}, t)$:

$$\dot{\boldsymbol{\varphi}}(\boldsymbol{x}, t) = -(\nabla^2 + \mu)\boldsymbol{\varphi}(\boldsymbol{x}, t) + \sqrt{g}\boldsymbol{\varphi}(\boldsymbol{x}, t)\sigma(\boldsymbol{x}, t) + \eta_\varphi(\boldsymbol{x}, t) \tag{2.5}$$

$$\dot{\sigma}(\boldsymbol{x}, t) = -\sigma(\boldsymbol{x}, t) + \frac{\sqrt{g}}{2}\boldsymbol{\varphi}^2(\boldsymbol{x}, t) + \eta_\sigma(\boldsymbol{x}, t) \tag{2.6}$$

with two independent thermal noises η_φ, η_σ.

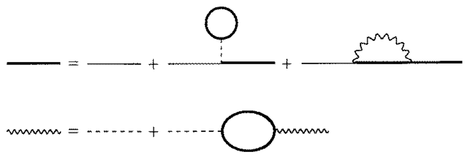

Figure 1. Diagrammatic equations for the correlation functions $\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}')\rangle$ (full line) and $\langle\sigma(\boldsymbol{x})\sigma(\boldsymbol{x}')\rangle$ (wavy line).

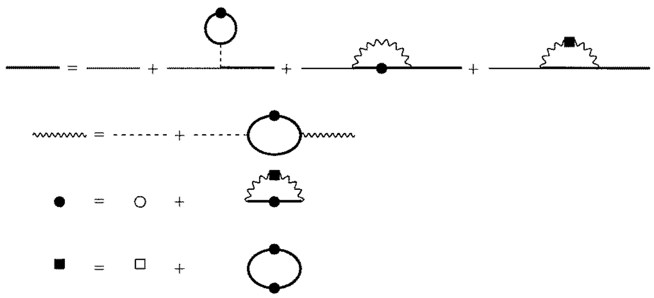

Figure 2. Diagrammatic representation of the dynamical SCSA equations, where the full circle stands for the renormalization of D_φ while the full square stands for the renormalization of D_σ. The empty circle and empty square stand for the non-renormalized noises.

Let us now consider the two-point functions

$$G_\varphi(\boldsymbol{x},\boldsymbol{x}',t,t') = \left\langle \frac{\partial\varphi(\boldsymbol{x},t)}{\partial\eta(\boldsymbol{x},t')}\right\rangle \tag{2.7}$$

$$C_\varphi(\boldsymbol{x},\boldsymbol{x}',t,t') = \langle\varphi(\boldsymbol{x},t)\varphi(\boldsymbol{x}',t')\rangle \tag{2.8}$$

and the corresponding functions for the field σ. The SCSA dynamical equations, which can be seen as a mode-coupling approximation on the set of equations (2.5)–(2.6) (see figure 1) then read:

$$\begin{aligned}\Sigma_\varphi(t_1,t_2) = {} & n\frac{g}{2}\delta(t_1,t_2)\int_0^{t_1}\mathrm{d}t_3\, C_\varphi(t_3,t_3)G^0_\sigma(t_1,t_3)\\ & +g[G_\varphi(t_1,t_2)C_\sigma(t_1,t_2)+G_\sigma(t_1,t_2)C_\varphi(t_1,t_2)]\end{aligned} \tag{2.9}$$

$$\Sigma_\sigma(t_1,t_2) = ngG_\varphi(t_1,t_2)C_\varphi(t_1,t_2) \tag{2.10}$$

$$D_\varphi(t_1,t_2) = 2T\delta(t_1-t_2) + gC_\varphi(t_1,t_2)C_\sigma(t_1,t_2) \tag{2.11}$$

$$D_\sigma(t_1,t_2) = 2T\delta(t_1,t_2) + n\frac{g}{2}C^2_\varphi(t_1,t_2) \tag{2.12}$$

where we have dropped the space coordinates, $\boldsymbol{x}$, for clarity, and introduced the self-energies, Σ, defined as:

$$G(t,t') = G^0(t,t') + \int_0^t \mathrm{d}t_1\int_{t'}^{t_1}\mathrm{d}t_2\, G^0(t,t_1)\Sigma(t_1,t_2)G(t_2,t') \tag{2.13}$$

(the superior 0 refers to the bare quantity), and the 'renormalized noises' D, defined as:

$$C(t,t') = \int_0^t \mathrm{d}t_1\int_0^{t'}\mathrm{d}t_2\, G(t,t_1)D(t_1,t_2)G(t',t_2). \tag{2.14}$$

We shall limit ourselves to consider the above equations in a regime of stationary dynamics. That is to say that we will make use of the assumption of time-translational invariance (only difference of times matter), which allows one to show that the fluctuation dissipation theorem (FDT) is valid, i.e.

$$\theta(t-t')\frac{\partial C(t-t')}{\partial t'} = TG(t-t'). \tag{2.15}$$

Extensions of these methods to the non-stationary low-temperature regime, where this theorem is violated [11], will be the subject of further work. In the following, we shall set the energy scales by choosing $T = 1$, and vary the mass term μ to reach the critical point.

3. Static limit

With these assumptions equations (2.12) reduces to only two coupled independent equations which have the simplest form in Fourier space

$$\begin{aligned}\Sigma_\varphi(k,\omega) = g\int[C_\sigma(k-k',\omega-\omega')G_\varphi(k',\omega') + C_\varphi(k-k',\omega-\omega')G_\sigma(k',\omega')]\mathrm{D}k'\mathrm{D}\omega' \\ +\frac{ng}{2}G^0_\sigma(k=0,\omega=0)\int C_\varphi(k',\omega')\mathrm{D}k'\mathrm{D}\omega' \end{aligned}\tag{3.1}$$

$$\Sigma_\sigma(k,\omega) = ng\int C_\varphi(k-k',\omega-\omega')G_\varphi(k',\omega')\mathrm{D}k'\mathrm{D}\omega' \tag{3.2}$$

where $\mathrm{D}k' \equiv \frac{\mathrm{d}^d k'}{(2\pi)^d}$ and $\mathrm{D}\omega' \equiv \frac{\mathrm{d}\omega'}{2\pi}$.

Using the fact that $C(k, t = 0) \equiv \mathcal{C}(k)$ is equal to $G(k, \omega = 0)$ (from FDT and the Kramers–Kronig (KK) relations), and using again the KK relations, it is easy to check that for $\omega = 0$ one recovers exactly the static SCSA equations [5], namely

$$\begin{aligned}\mathcal{C}_\varphi(k) &= \frac{1}{\mu + k^2 - g\int \mathrm{D}k'\mathcal{C}_\varphi(k-k')\mathcal{C}_\sigma(k') - \frac{gn}{2}\int \mathrm{D}k'\mathcal{C}_\varphi(k')} \\ \mathcal{C}_\sigma(k) &= \frac{1}{1-\frac{g}{2}n\int \mathrm{D}k'\mathcal{C}_\varphi(k-k')\mathcal{C}_\varphi(k')}.\end{aligned}\tag{3.3}$$

In order to test the validity of this approximation, it is interesting to consider the case of zero spatial dimensions [6]. Let us set $n = 1$ which is a bad case for the SCSA which should become more accurate the larger n is. We will compare equations (3.3) with the exact static-correlation function which, in zero dimension, can be calculated analytically and is

$$\mathcal{C}_{\text{exact}} = -\frac{1}{\mu} + \frac{\mu}{g} - \frac{\mu K_{-\frac{3}{4}}\left(\frac{\mu^2}{4g}\right)}{2gK_{\frac{1}{4}}\left(\frac{-\mu^2}{4g}\right)} - \frac{\mu K_{\frac{5}{4}}\left(\frac{\mu^2}{4g}\right)}{2gK_{\frac{1}{4}}\left(\frac{-\mu^2}{4g}\right)} \tag{3.4}$$

where $K_n(a)$ is the modified Bessel function of the second kind. Equations (3.3) give for $\mathcal{C}_\varphi$:

$$\mathcal{C}_{\text{SCSA}} = \frac{1}{\left(\mu - n\frac{g}{2}\mathcal{C}_{\text{SCSA}} - g\frac{\mathcal{C}_{\text{SCSA}}}{(1-\frac{g}{2}n\mathcal{C}^2_{\text{SCSA}})}\right)}. \tag{3.5}$$

From plotting the relative difference of the two correlation functions versus the coupling (see figure 3) we can see that SCSA is quite close to the exact theory. In particular, the asymptotic behaviour in the $|g| \to \infty$ limit of the two functions is

$$\lim_{|g|\to\infty}\sqrt{|g|}\mathcal{C}_{\text{SCSA}} = 2(\sqrt{2}-1) \qquad \text{and} \qquad \lim_{|g|\to\infty}\sqrt{|g|}\mathcal{C}_{\text{exact}} = \frac{2\sqrt{2}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}. \tag{3.6}$$

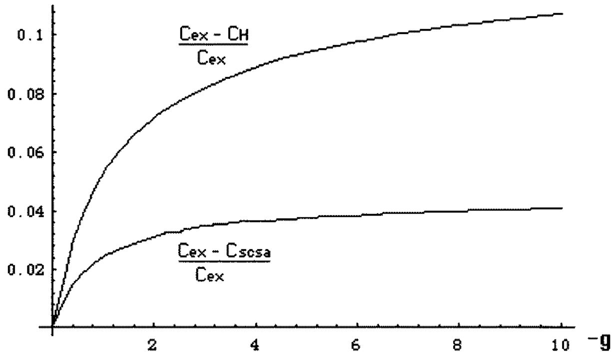

Figure 3. Relative difference between the exact result and the Hartree ($\mathcal{C}_{\rm H}$) and the SCSA ($\mathcal{C}_{\rm SCSA}$) approximations, in the case $n = 1, d = 0$.

For all g, the relative difference is actually bounded by:

$$\frac{|\mathcal{C}_{\rm exact} - \mathcal{C}_{\rm SCSA}|}{\mathcal{C}_{\rm exact}} < 1 - \frac{\sqrt{-1+\sqrt{2}}\,\Gamma(\frac{1}{4})}{2\Gamma(\frac{3}{4})} = 0.0479\ldots \tag{3.7}$$

We can also compare the small g expansions of the two theories which give

$$\mathcal{C}_{\rm exact} = \frac{1}{\mu}\left(1 + \frac{3}{2\mu^2}g + \frac{21}{4\mu^4}g^2\right) \tag{3.8}$$

$$\mathcal{C}_{\rm SCSA} = \frac{1}{\mu}\left(1 + \frac{3}{2\mu^2}g + \frac{5}{\mu^4}g^2\right) \tag{3.9}$$

showing explicitly how the two theories differ already at order g^2. The self-consistent nature of the approximation, however, keeps the SCSA in good agreement with the exact theory even for large values of the coupling constant as remarked before.

It is instructive, in passing, to compare the SCSA with the simple Hartree ($n = \infty$) resummation scheme, which is also the Gaussian variational result. One defines $F_H = \min\{F\}$ where

$$F = F_0 + \langle H - H_0 \rangle \tag{3.10}$$

with

$$F_0 = -\ln \int \mathcal{D}\varphi \, {\rm e}^{-\frac{\tilde{\mu}\varphi^2}{2}} = -\ln\left(\frac{2\pi}{\tilde{\mu}}\right) \tag{3.11}$$

$$\langle H_0 \rangle = \tfrac{1}{2} \tag{3.12}$$

$$\langle H \rangle = \int \mathcal{D}\varphi \, {\rm e}^{-\frac{\tilde{\mu}\varphi^2}{2}} \left(\frac{\mu}{2}\varphi^2 - \frac{g}{8}\varphi^4\right) = \left(\frac{\mu}{2\tilde{\mu}} - \frac{3g}{8\tilde{\mu}^2}\right). \tag{3.13}$$

Minimizing F with respect to $\tilde{\mu}$ we find

$$\mu_H = \frac{\mu + \sqrt{\mu^2 - 6g}}{2} \tag{3.14}$$

and consequently

$$\mathcal{C}_H \langle \varphi^2 \rangle_{\mu_H} = \frac{2}{\mu + \sqrt{\mu^2 - 6g}}. \tag{3.15}$$

As can be seen from figure 3, the SCSA turns out to be marginally better than the Hartree variational approach (at least in this particular case of $n = 1$ and $d = 0$).

4. Critical dynamics

We shall now work right at the critical point, μ_c, such that the renormalized mass vanishes (therefore eliminating the 'tadpole' contribution in equation 2.9). We shall search for solutions under the general dynamic scaling form (valid in the small k and small ω limit):

$$G_\varphi(k,\omega) = \frac{1}{k^\Delta} n_\varphi\left(\frac{\omega}{k^z}\right) \qquad G_\sigma(k,\omega) = \frac{1}{k^{\Delta'}} n_\sigma\left(\frac{\omega}{k^z}\right)$$
$$C_\varphi(k,\omega) = \frac{2}{\omega k^\Delta}\,\mathrm{Im}\left[n_\varphi\left(\frac{\omega}{k^z}\right)\right] \qquad C_\sigma(k,\omega) = \frac{2}{\omega k^{\Delta'}}\,\mathrm{Im}\left[n_\sigma\left(\frac{\omega}{k^z}\right)\right] \tag{4.1}$$

where we have defined $\Delta = 2-\eta$, and used FDT. First setting $\omega = 0$, one finds by matching the momentum dependence of the left- and right-hand sides of (3.1)–(3.2) that:

$$\Delta' = d - 2\Delta = d - 4 + 2\eta. \tag{4.2}$$

Note that in the mean field, $z = 2$, $\Delta = 2$, $\eta = 0$ and $\Delta' = 0$. Identification of the prefactors yields:

$$n_\sigma(0)n_\varphi^2(0) = -\frac{2}{f(\eta,d)ng} \tag{4.3}$$

where

$$f(\eta,d) = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left[\Delta - \frac{d}{2}\right]\Gamma\left[\frac{d-\Delta}{2}\right]^2}{\Gamma[d-\Delta]\Gamma\left[\frac{\Delta}{2}\right]^2} \tag{4.4}$$

and an extra equation fixing η as a function of d and n, which we do not write explicitly [5].

Now let us consider the other case where $k = 0$ and $\omega > 0$ (but small). Taking the imaginary part of (3.1)–(3.2), one obtains:

$$\mathrm{Im}[\Sigma_\varphi(0,\omega)] = \frac{S\omega}{nn_\varphi(0)}\int q^{\Delta-1}\,\mathrm{d}q\,\mathrm{d}s\,\frac{\mathrm{Im}\left[f_\varphi\left(\frac{(\omega-s)}{q^z}\right)\right]\mathrm{Im}\left[f_\sigma\left(\frac{s}{q^z}\right)\right]}{s(\omega-s)} \tag{4.5}$$

$$\mathrm{Im}[\Sigma_\sigma(0,\omega)] = \frac{S}{n_\sigma(0)}\int q^{\Delta'-1}\,\mathrm{d}q\,\mathrm{d}s\,\frac{\mathrm{Im}\left[f_\varphi\left(\frac{(\omega-s)}{q^z}\right)\right]\mathrm{Im}\left[f_\varphi\left(\frac{(s)}{q^z}\right)\right]}{s} \tag{4.6}$$

where $f_{\varphi,\sigma}(x) = n_{\varphi,\sigma}(x)/n_{\varphi,\sigma}(0)$. We also defined

$$S = \frac{2ng\Omega_d}{(2\pi)^{(d+1)}}n_\varphi^2(0)n_\sigma(0) \equiv -\frac{4\Omega_d}{f(\eta,d)(2\pi)^{(d+1)}}. \tag{4.7}$$

In general the scaling functions can be written

$$\mathrm{Im}[f_\varphi(x)] \doteq A\tilde{f}_\varphi(ax)$$
$$\mathrm{Im}[f_\sigma(x)] \doteq A'\tilde{f}_\sigma(a'x) \tag{4.8}$$

with, by convention, $\lim_{u\to\infty} u^{\Delta/z}\tilde{f}_\varphi(u) = 1$ and $\lim_{u\to\infty} u^{\Delta'/z}\tilde{f}_\sigma(u) = 1$. This asymptotic behaviour is required for the $k \to 0$ limit to be well defined, if (4.1) is correct. Furthermore, the small ω behaviour of the imaginary part of the response function is expected to be regular for k finite, and hence $\tilde{f}(u) \propto u$ for $u \to 0$. A, A' are coefficients setting the scale of the imaginary part of the response function while a, a' are coefficients setting the frequency scales. Using the fact that the imaginary and real part of the response function are power-laws at large frequencies, which imply that their ratio is $\tan\left(\frac{\pi\Delta}{2z}\right)$ (resp. $\tan\left(\frac{\pi\Delta'}{2z}\right)$) one finds that:

$$\frac{a^{\Delta/z}}{A}\sin^2\left(\frac{\pi\Delta}{2z}\right) = \frac{S}{nz}\int_0^\infty \frac{\mathrm{d}x}{x^{1+\Delta/z}}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{u(1-u)}\,\mathrm{Im}[f_\varphi(x(1-u))]\,\mathrm{Im}[f_\sigma(xu)]$$
$$\frac{\alpha'^{\Delta'/z}}{A'}\sin^2\left(\frac{\pi\Delta'}{2z}\right) = \frac{S}{z}\int_0^\infty \frac{\mathrm{d}x}{x^{1+\Delta'/z}}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{u}\,\mathrm{Im}[f_\varphi(x(1-u))]\,\mathrm{Im}[f_\varphi(xu)]. \tag{4.9}$$

It is easy to show that these equations actually only depend on the value of the *ratio* of frequency scales $y = \frac{a'}{a}$. The coefficient A can be fixed using the KK relation, since the involved integral converges, which means that the small k behaviour of the real part of the correlation function is fully determined by the imaginary part in the scaling region $\omega, k \to 0$. Hence

$$1 = \frac{A}{\pi}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{\tilde{f}_\varphi(x)}{x}. \tag{4.10}$$

The corresponding integral for $\tilde{f}_\sigma$ does not converge for large x, meaning that the non-scaling region is needed to saturate the sum rule. Hence, we must use another relation to fix A', which we choose to be the small ω expansion of equation (4.6).

Thus, *if* the functions $\tilde{f}_\varphi$, $\tilde{f}_\sigma$ were known, we would have four equations to fix four constants: A, A', y, and, of course, the dynamical exponent z, in terms of d and n. $\tilde{f}_\varphi$, $\tilde{f}_\sigma$ are in principle also fixed by the full equations for arbitrary $\frac{\omega}{k^z}$. However, as in other similar cases [9], these equations are very hard to solve, either analytically or numerically. We will thus propose the ansatz for these functions, which have to satisfy the above general requirements. Note that once A, A', a, a' have been pulled out, the only freedom is in the *shape* of these functions. We shall thus work with two such ansatz, which will turn out to give very similar answers for z. This was also the case in the context of the KPZ equation [9].

5. Ansatz 1

The simplest ansatz one can think of, which generalizes the mean-field shape:

$$\tilde{f}_\varphi(x) = \frac{x}{(1+x^2)} \tag{5.1}$$

reads:

$$\tilde{f}_\varphi(x) = \frac{x}{(1+x^2)^\alpha} \tag{5.2}$$

$$\tilde{f}_\sigma(x) = \frac{x}{(1+x^2)^{\alpha'}} \tag{5.3}$$

where we have set

$$\alpha \doteq \frac{\Delta+z}{2z} \tag{5.4}$$

$$\alpha' \doteq \frac{\Delta'+z}{2z}. \tag{5.5}$$

(Note that $\alpha = 1$ in mean field.) These functions indeed have the correct asymptotic behaviours; they go linearly to zero for small values of the argument and behave as power-laws ($\tilde{f}_\varphi(x) \simeq x^{-\frac{\Delta}{z}}$ and $\tilde{f}_\sigma(x) \simeq x^{-\frac{\Delta'}{z}}$) in the large x limit.

We can now use (4.10) to determine A

$$A = \sqrt{\pi}\,\frac{\Gamma[\alpha]}{\Gamma[\alpha - \frac{1}{2}]}. \tag{5.6}$$

The small ω expansion of $\mathrm{Im}\,\Sigma_\sigma(k,\omega)$ can be matched with that of the right-hand side of equation (4.6) leading to the following equation

$$y = -\frac{2A^2}{A' f(\eta,d)(2\pi)^{d+1}} \int_{-\infty}^{\infty} \mathrm{d}^d q\, \frac{1}{|q|^{\Delta}|1-q|^{\Delta+z}} \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{1}{(1+t^2)^{\alpha}\left(1+\left(\frac{|q|^z t}{|1-q|^z}\right)^2\right)^{\alpha}}. \tag{5.7}$$

After some algebraic manipulations we obtain for the last three equations:

$$\sin^2\left(\frac{\pi\Delta}{2z}\right) = -\frac{A^2 A' y S}{2nz} B\left[1-\frac{\Delta}{2z}, \frac{d}{2z}\right] \times \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{|u|^{2-\frac{\Delta}{z}}} F\left[\alpha', 1-\frac{\Delta}{2z}, \alpha+\alpha', 1-y^2\frac{(1-u)^2}{u^2}\right] \tag{5.8}$$

$$\sin^2\left(\frac{\pi\Delta'}{2z}\right) = -\frac{A^2 A' S}{2z} y^{-\frac{\Delta'}{z}} B\left[1-\frac{\Delta'}{2z}, \frac{d}{2z}\right] \times \int_{-\infty}^{\infty} \frac{u\,\mathrm{d}u}{|u|^{2-\frac{\Delta'}{z}}} F\left[\alpha, 1-\frac{\Delta'}{2z}, 2\alpha, \frac{2u-1}{u^2}\right] \tag{5.9}$$

$$y = \pi\frac{A^2 S}{z A' \Omega_d} B\left[\frac{1}{2}, 2\alpha-\frac{1}{2}\right] \times \int_0^{\infty} \mathrm{d}q\, q^{d-2-\Delta} \int_{|1-q|^{2z}}^{|1+q|^{2z}} \frac{\mathrm{d}x}{x^{\frac{\Delta+3z-2}{2z}}} F\left[\alpha, \frac{1}{2}, 2\alpha, 1-\frac{q^{2z}}{x}\right] \tag{5.10}$$

where $B[a,b]$ and $F[a,b,c,x]$ are the Euler beta and hypergeometric functions and where the last equation (5.10) was written for the special case $d = 3$ which we shall consider below. We can solve analytically equations (5.7)–(5.9) at order ϵ^2 to compare with the exact RG treatment of [10]. At lowest order we obtain:

$$c = \frac{8\ln 2}{\pi}\frac{\arctan\sqrt{\frac{1-y^2}{y^2}}}{\sqrt{1-y^2}} - 1 \tag{5.11}$$

$$A' = -\frac{\pi\epsilon}{4} \tag{5.12}$$

$$y = \frac{4\ln 2}{\pi} \tag{5.13}$$

where we have defined, following [10],

$$z = 2 + c\eta. \tag{5.14}$$

The order $\mathrm{O}(\epsilon^2)$ RG results reads, $c = 6\ln\frac{4}{3} - 1 = 0.7261$. The form (5.14) means that to lowest-order z depends on n only through the static exponent η. On the other hand, equations (5.13) give

$$c = 0.8376 \tag{5.15}$$

in slight disagreement with the exact result. This comes from the fact that while our ansatz for $\tilde{f}_\varphi$ is exact in the limit $\epsilon \to 0$, the corresponding ansatz for $\tilde{f}_\sigma$ is already wrong at lowest order since it does not satisfy equation (4.6). In our second ansatz, we thus keep the same shape for $\tilde{f}_\varphi$, but choose, for $\tilde{f}_\sigma$, a form which is exact when $\epsilon \to 0$.

6. Ansatz 2

Knowing the mean-field form for $f_\varphi(x)$ we can, at lowest order in ϵ, write for $\mathrm{Im}[f_\sigma(x)]$

$$\mathrm{Im}\, f_\sigma(x) = 2^{d-4}\frac{f(\eta,d)}{\pi^{d/2}}\frac{(2\pi)^d}{\Gamma\left[2-\frac{d}{2}\right]}\,\mathrm{Im}\left[\frac{1}{\xi(x)}\right] \tag{6.1}$$

where

$$\xi(x) = 1 - \frac{\epsilon}{2}\int_0^1 \mathrm{d}t\, \log[1-t^2-2\mathrm{i}x(1-t)]. \tag{6.2}$$

It is then straightforward to generalize $\mathrm{Im}[f_\sigma(x)]$ to general dimensions as:

$$\tilde{f}_\sigma \propto \mathrm{Im}[(2-\mathrm{i}x)^{1-\frac{\Delta'}{z}} - (1-\mathrm{i}x)^{1-\frac{\Delta'}{x}}] \tag{6.3}$$

with a prefactor ensuring that the coefficient of $x^{-\frac{\Delta'}{z}}$ for large x is unity. Equation (5.8) is now replaced by

$$\sin^2\left(\frac{\pi\Delta}{2z}\right) = \frac{A^2A'Sb}{nz}\int_0^\infty \frac{\mathrm{d}r}{r^{\frac{\Delta}{z}}} \times \int_{-\infty}^{\infty}\mathrm{d}u\,\frac{\mathrm{Im}\left[\left(2-\mathrm{i}\frac{\pi}{2\ln 2}(yru)\right)^{1-\frac{\Delta'}{z}} - \left(1-\mathrm{i}\frac{\pi}{2\ln 2}(yru)\right)^{1-\frac{\Delta'}{z}}\right]}{u[1+r^2(1-u)^2]^\alpha} \tag{6.4}$$

where now b is given by:

$$b = \frac{2\ln 2}{\pi\left(2^{-\frac{\Delta'}{z}}-1\right)\left(\frac{\Delta'}{z}-1\right)}. \tag{6.5}$$

We finally obtain a set of equations for z of the same kind as (5.8)–(5.10) but which are now exact up to $\mathrm{O}(\epsilon^2)$, as we have checked directly.

7. Numerical results

We solved numerically both sets of equations in $d = 3$ for $n = 1, \ldots, 10$. We used the values of $\eta(d = 3, n)$ that can be derived from the formula reported in [5]. The values obtained for z are reported in table 1.

As it was hoped, the results are fairly independent from the ansatz used, which is more and more true for large n. The result for $n = 1$ is rather close to the best Monte Carlo

Table 1.

n	z (ansatz 1)	z (ansatz 2)
1	2.119	2.113
2	2.071	2.069
3	2.050	2.049
4	2.038	2.038
5	2.031	2.031
6	2.0258	2.0258
7	2.0223	2.0222
8	2.0196	2.0195
9	2.0174	2.0174
10	2.0157	2.0157

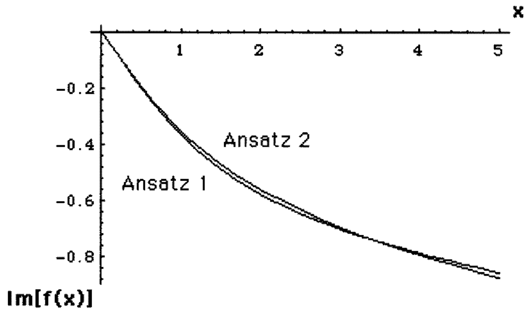

Figure 4. The two ansatz for the functions $f_\sigma(x)$, $n = 1$.

estimate of [12], which gives $z = 2.09 \pm 0.02$. Let us note, however, that the SCSA overestimates significantly η in $d = 3$.

In figure 4 we compare the two different choices for the scaling function $f_\sigma(x)$ with their relative values of the parameters y, A' and z, and in the case $n = 1$, $d = 3$. We notice that the constraints for small x and large x very much restrict the freedom on the shape of this function.

Finally, a linear regression of our results for $n = 1$–10 gives $z \simeq 2 + c\eta$ with $c = 0.64$, which is lower than the $\mathrm{O}(\epsilon^2)$ result, but larger than the exact result for $\mathrm{d} = 3, n \to \infty$, i.e. $c = \frac{1}{2}$ [10].

8. Conclusions

The aim of this paper was to extend the static SCSA to dynamics, in particular the properties of the critical dynamics of the ϕ^4 model. Although the resulting equations cannot be fully solved, a much-constrained ansatz leads to a value of the exponent z in rather good agreement with Monte Carlo data.

Our work was originally inspired by glassy dynamics: the SCSA equations actually describe in exactly the dynamics of some mean-field spin-glass-like models. It would be interesting to study these equations in the low-temperature phase, where dynamics becomes non-stationary (aging) and FDT is lost. For ϕ^4 models, this corresponds to a coarsening regime [1]. It would be interesting to know whether the SCSA equations properly describe this regime, and can compete with other approximation schemes [1, 13].

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